INCOMPLETE BLOCK AND LATTICE RECTANGLE DESIGNS FOR v = 36 USING F-SQUARE THEORY

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Abstract

Using F-square theory, it was possible to construct several new designs. Incomplete block design plans with incomplete block sizes k = 2, 3, 4, 9, 12, and 18 with 3, 4, 5, 5, 10, and 11 confounding arrangements or plans were obtained. Various lattice rectangle design plans were also obtained. Two-row by 18-column lattice rectangle designs with three arrangements were constructed. A three-row by 12-column lattice rectangle design with two arrangements was obtained. Also, two arrangements of a 4-row by 9-column lattice rectangle dsign was found. Statistical analyses for these designs follows standard analyses found in the literature.

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Introduction

Recent work on orthogonal F-square designs for squares of side six has been substantial. From the results of Finney (1982) and Kirton (1984), it would appear that all orthogonal sets of F-squares have been obtained. In the literature, several incomplete block and row-column plans have been obtained for $v = 36$ treatments for various block sizes k. For $k = 6$, double and triple lattice design plans have been constructed by using the rows, the columns, and the letters of a latin square design plan of order six to determine the incomplete blocks for three arrangements. If these confounding arrangements of rows, columns, and letters are designated as A, B, and C, respectively, then for $r = 4$ arrangements, one may duplicate any one of the confounding arrangements A, B, or C; for $r = 5$, any two of the three arrangements may be duplicated; and for $r = 6$, all three arrangements would be duplicated. Larger values of r may be obtained by using triplicates, etc. Alternatively, the procedures of Patterson and Williams (1976) or of Khare and Federer (1981) may be used to obtain plans.

For a 6-row by 6-column plan, the following confounding schemes may be used (see Kempthorne and Federer, 1948, and Federer, 1950) to obtain three lattice square arrangements I, II, and III:

	Arrangement		
	I	II	III
Effect confounded with rows	A	C	B
Effect confounded with columns	B	A	C

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Each effect is unconfounded in one of the three arrangements which results in a connected design. Additional arrangements (complete blocks) may be obtained by duplication of arrangements.

Another approach is to use factorial confounding theory (see, e.g., Kempthorne, 1952, and Federer, 1955) to obtain blocks of size $k = 6, 9, 12$, or 18. If the $v = 36$ treatments are designated as a $2^2 3^2$ factorial with factor W at two levels, factor X at two levels, factor Y at three levels, and factor Z at three levels, W, X, and WX effects may be confounded to obtain three plans for block size $k = 18$. In addition, one could confound, for example, WXY_Q, WXZ_Q, $WXY_Q Z_Q$, $WY_Q Z_Q$, and $XY_Q Z_Q$, to obtain five additional arrangements for $k = 18$. For $k = 12$, Y could be confounded with incomplete blocks in arrangement I, Z in arrangement II, YZ in arrangement III, and YZ^2 in arrangement IV to obtain four different confounding arrangements. To obtain blocks of size $k = 9$, W, X, and WX could be completely confounded; or, W, X, and WX confounded in plan I and WXY_Q, WXZ_Q, and their interaction could be confounded for plan II. Other schemes are possible. For $k = 6$, one could confound X and Y in plan I, X and Z in plan II, W and Y in plan III, and W and Z in plan IV. Use of confounding schemes given by Yates (1937), Kempthorne (1952), and Federer (1955) results in other confounding schemes and arrangements.

Various row-column plans, e.g., 2×18, 3×12, and 4×9, may be obtained using complete confounding of various effects. Additional rectangles may be obtained by confounding other effects in other plans. This would avoid complete confounding of any one effect if more than one row-column arrangement were used.

Our purpose here is to demonstrate how to construct various incomplete block and row-column designs using orthogonal F-square and F-rectangle design theory. New plans are obtained in some cases.

Construction of Incomplete Block Designs

From the works of Federer (1975), Finney (1982), and Kirton (1984), a considerable amount of F-square theory is available. For example, it is known that eight pairwise orthogonal F-squares of order six, with three symbols appearing twice in each row and twice in each column, denoted as POFS(6;2,2,2;8), exist. If one uses the three symbols in an F-square to determine the three incomplete blocks of size k = 12, eight arrangements result. In addition, one arrangement may be obtained from rows and a second from columns. To demonstrate, we obtain the eight F-squares from the above authors and write the numbers 1 to 36 in the same order in each F-square. These are given in Table 1. Thus, for FS_1 there are three symbols; numbers 1, 5, 7, 8, 17, 18, 20, 21, 28, 30, 33, and 34 appear with symbol 1, twelve different numbers (4, 6, 9, 12, 13, 15, 22, 23, 26, 29, 31, and 32) appear with symbol 2, and the remaining twelve numbers appear with symbol 3. This results in arrangement 1. In a similar manner, seven additional arrangements may be obtained from FS_2 to FS_8 (see Table 2). Arrangements 9 and 10 were obtained from rows and columns, respectively.

From Finney (1982) and Kirton (1984), we note that a POFS(6;3,3;9) set exists. This set allows construction of nine arrangements in block sizes of k = 18; two additional arrangements are obtained from rows and columns. Note that from factorial theory we may obtain twelve arrangements (i.e., W, X, WX, WY_Q, WZ_Q, XY_Q, XZ_Q, WXY_Q, WXZ_Q, WY_QZ_Q, XY_QZ_Q, and WXY_QZ_Q). It was not ascertained if ten POFS(6;3,3)s result from these arrangements.

Table 1. A POFS(6;2,2,2;8) set together with numbers 1 to 36 = v treatments. (Finney, 1982, p. 145)

FS_1:	1	1	1	7	2	13	3	19	3	25	2	31
	3	2	1	8	3	14	1	20	2	26	2	32
	3	3	2	9	2	15	1	21	3	27	1	33
	2	4	3	10	3	16	2	22	1	28	1	34
	1	5	3	11	1	17	2	23	2	29	3	35
	2	6	2	12	1	18	3	24	1	30	3	36
FS_2:	1	1	2	7	1	13	2	19	3	25	3	31
	1	2	3	8	3	14	2	20	1	26	2	32
	2	3	2	9	3	15	1	21	1	27	3	33
	3	4	3	10	2	16	1	22	2	28	1	34
	3	5	1	11	1	17	3	23	2	29	2	35
	2	6	1	12	2	18	3	24	3	30	1	36
FS_3:	1	1	2	7	2	13	3	19	1	25	3	31
	3	2	3	8	2	14	1	20	1	26	2	32
	1	3	3	9	1	15	3	21	2	27	2	33
	3	4	1	10	3	16	2	22	2	28	1	34
	2	5	2	11	3	17	1	23	3	29	1	35
	2	6	1	12	1	18	2	24	3	30	3	36
FS_4:	1	1	2	7	3	13	1	19	3	25	2	31
	3	2	1	8	2	14	3	20	2	26	1	32
	2	3	3	9	1	15	2	21	1	27	3	33
	2	4	3	10	1	16	3	22	2	28	1	34
	3	5	1	11	2	17	1	23	3	29	2	35
	1	6	2	12	3	18	2	24	1	30	3	36

Table 1 (continued).

FS_5:	1	1	2	7	3	13	3	19	2	25	1	31
	1	2	2	8	3	14	3	20	2	26	1	32
	2	3	3	9	1	15	1	21	3	27	2	33
	3	4	1	10	2	16	2	22	1	28	3	34
	3	5	1	11	2	17	2	23	1	29	3	35
	2	6	3	12	1	18	1	24	3	30	2	36

FS_6:	1	1	2	7	3	13	3	19	2	25	1	31
	2	2	3	8	1	14	1	20	3	26	2	32
	1	3	2	9	3	15	3	21	2	27	1	33
	2	4	3	10	1	16	1	22	3	28	2	34
	3	5	1	11	2	17	2	23	1	29	3	35
	3	6	1	12	2	18	2	24	1	30	3	36

FS_7:	1	1	2	7	3	13	3	19	2	25	1	31
	3	2	1	8	2	14	2	20	1	26	3	32
	3	3	1	9	2	15	2	21	1	27	3	33
	2	4	3	10	1	16	1	22	3	28	2	34
	1	5	2	11	3	17	3	23	2	29	1	35
	2	6	3	12	1	18	1	24	3	30	2	36

FS_8^*:	1	1	1	7	2	13	2	19	3	25	3	31
	1	2	2	8	1	14	3	20	2	26	3	32
	2	3	1	9	1	15	3	21	3	27	2	33
	2	4	3	10	3	16	1	22	1	28	2	34
	3	5	2	11	3	17	1	23	2	29	1	35
	3	6	3	12	2	18	2	24	1	30	1	36

* A,B = 1; C,D = 2; and E,F = 3.

Table 2. Ten arrangements for incomplete blocks of size k = 12.

Arrangement and Symbol

1			2			3			4			5		
1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
1	4	2	1	3	4	1	5	2	1	3	2	1	3	4
5	6	3	2	6	5	3	6	4	6	4	5	2	6	5
7	9	10	11	7	8	10	7	8	8	7	9	10	7	9
8	12	11	12	9	10	12	11	9	11	12	10	11	8	12
17	13	14	13	16	14	15	13	16	15	14	13	15	16	13
18	15	16	17	18	15	18	14	17	16	17	18	18	17	14
20	22	19	21	19	23	20	22	19	19	21	20	21	22	19
21	23	24	22	20	24	23	24	21	23	24	22	24	23	20
28	26	25	26	28	25	25	27	29	27	26	25	28	25	27
30	29	27	27	29	30	26	28	30	30	28	29	29	26	30
33	31	35	34	32	31	34	32	31	32	31	33	31	33	34
34	32	36	36	35	33	35	33	36	34	35	36	32	36	35

6			7			8			9			10		
1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
1	2	5	1	4	2	1	3	5	1	3	5	1	13	25
3	4	6	5	6	3	2	4	6	2	4	6	2	14	26
11	7	8	8	7	10	7	8	10	7	9	11	3	15	27
12	9	10	9	11	12	9	11	12	8	10	12	4	16	28
14	17	13	16	14	13	14	13	16	13	15	17	5	17	29
16	18	15	18	15	17	15	18	17	14	16	18	6	18	30
20	23	19	22	20	19	22	19	20	19	21	23	7	19	31
22	24	21	24	21	23	23	24	21	20	22	24	8	20	32
29	25	26	26	25	28	28	26	25	25	27	29	9	21	33
30	27	28	27	29	30	30	29	27	26	28	30	10	22	34
31	32	35	31	34	32	35	33	31	31	33	35	11	23	35
33	34	36	35	36	33	36	34	32	32	34	36	12	24	36

The following incomplete block design plans may be constructed from sets of orthogonal F-squares:

Block size k	Sets used	Number of arrangements
2	POFS(6;2,2,2;8) and POFS(6;3,3;8)	3
3	POFS(6;2,2,2;8) and POFS(6;3,3;8)	4
4	POFS(6;2,2,2;8)	5
6	Latin square of order 6	3
9	POFS(6;3,3;8)	5
12	POFS(6;2,2,2;8)	10
18	POFS(6;3,3;9)	11

The method for constructing plans for k = 6, 12, and 18 has been described above. For k = 9, one takes pairs of plans for k = 18; there will be nine of the 18 treatments (numbers) in common with each of the two incomplete blocks of size k = 18 in plan 2. The nine common numbers of one incomplete block of k = 18 are put in one block of size k = 9, the uncommon in a second block of size k = 9. The same procedure is followed for the second set of k = 18 numbers in plan 1. Five complete blocks with four incomplete blocks of size k = 9 result. Also, these may be obtained from the pairs 11, 12, 21, and 22 in Table 4. Similarly, for blocks of k = 4, one takes five pairs of plans from the ten plans for k = 12 in Table 1. The symbol 1 in FS_1 has four treatments in common with each of the symbols 1, 2, and 3 in FS_2. The common treatments form the incomplete blocks of size k = 4. These are given in Table 3.

For k = 3, it is necessary to have one FS(6;2,2,2) and two FS(6;3,3)s which are mutually orthogonal. $S_9(6;2,2,2)$ is formed by letting rows 1 and 2 be symbol 1, rows 3 and 4 be symbol 2, and rows 5 and 6 be symbol 3. This forms a 6 x 6 array of ones, twos, and threes, from which plan 9 in Table 2 was obtained. Similarly, $S_{10}(6;2,2,2)$ is formed from columns. If one uses $S_9(6;2,2,2)$ together

Table 3. Five arrangements for incomplete blocks of size k = 4.

Arrangements and symbols for pairs of FS

1			2			3			4			5		
11	12	13	11	12	13	11	12	13	11	12	13	11	12	13
1	7	5	1	3	10	1	2	10	1	8	5	1	13	25
17	18	8	15	12	18	11	18	15	9	18	16	2	14	26
21	20	30	23	26	20	29	24	21	22	24	27	7	19	31
34	28	33	34	35	25	31	32	28	35	26	31	8	20	32
21	22	23	21	22	23	21	22	23	21	22	23	21	22	23
12	6	4	6	7	5	3	7	6	7	4	6	3	15	27
13	9	15	11	14	13	16	17	8	14	11	20	4	16	28
22	29	23	27	24	22	22	23	26	15	29	21	9	21	33
26	32	31	32	28	33	33	25	36	36	34	25	10	22	34
31	32	33	31	32	33	31	32	33	31	32	33	31	32	33
2	3	10	8	4	2	12	4	5	2	3	10	5	17	29
11	16	14	16	17	9	14	9	13	23	13	12	6	18	30
27	19	24	19	21	29	20	27	19	28	19	17	11	23	35
36	35	25	30	31	36	30	34	35	30	33	32	12	24	36

Table 4. Pairs of OF(6;3,3)s (from Finney, 1982) with numbers 1 to 36 to produce incomplete blocks of size k = 9.

$FS_1(6;3,3)$ and $FS_2(6;3,3)$

11	1	11	7	11	13	22	19	22	25	22	31
12	2	21	8	12	14	22	20	11	26	21	32
22	3	22	9	21	15	11	21	12	27	11	33
21	4	12	10	22	16	21	22	11	28	12	34
12	5	22	11	21	17	11	23	21	29	12	35
21	6	11	12	12	18	12	24	22	30	21	36

$FS_3(6;3,3)$ and $FS_4(6;3,3)$

11	1	12	7	21	13	22	19	22	25	11	31
22	2	12	8	21	14	11	20	11	26	22	32
11	3	22	9	12	15	11	21	22	27	21	33
22	4	21	10	11	16	21	22	12	28	12	34
12	5	11	11	22	17	12	23	21	29	21	35
21	6	21	12	12	18	22	24	11	30	12	36

$FS_5(6;3,3)$ and $FS_6(6;3,3)$

11	1	11	7	22	13	12	19	22	25	21	31
21	2	22	8	12	14	12	20	21	26	11	32
12	3	11	9	22	15	11	21	21	27	22	33
12	4	22	10	21	16	21	22	12	28	11	34
22	5	21	11	11	17	22	23	11	29	12	35
21	6	12	12	11	18	21	24	12	30	22	36

$FS_7(6;3,3)$ and $FS_8(6;3,3)$

11	1	22	7	22	13	11	19	12	25	21	31
21	2	11	8	11	14	22	20	12	26	22	32
22	3	12	9	11	15	11	21	21	27	22	33
21	4	12	10	21	16	12	22	21	28	12	34
12	5	21	11	22	17	22	23	11	29	11	35
12	6	21	12	12	18	21	24	22	30	11	36

Table 4 (continued).

Rows and columns, $S_9(6;3,3)$ and $S_{10}(6;3,3)$

11	1	11	7	11	13	21	19	21	25	21	31
11	2	11	8	11	14	21	20	21	26	21	32
11	3	11	9	11	15	21	21	21	27	21	33
12	4	12	10	12	16	22	22	22	28	22	34
12	5	12	11	12	17	22	23	22	29	22	35
12	6	12	12	12	18	22	24	22	30	22	36

with $FS_1(6;3,3)$ and $FS_2(6;3,3)$ from Table 4, one arrangement in blocks of size $k = 3$ is formed. A second arrangement is formed from $S_{10}(6;2,2,2)$ and $FS_3(6;3,3)$ and $FS_4(6;3,3)$; a third arrangement is formed from $FS_5(6;2,2,2)$, $FS_6(6;3,3)$, and $FS_7(6;3,3)$; and a fourth arrangement is obtained by using $S_9(6;3,3)$, formed from rows, $S_{10}(6;3,3)$, formed from columns, and $FS_1(6;2,2,2)$. These four sets of triplets are mutually orthogonal. It may be that more triplets of orthogonal squares can be found from other sets of pairwise orthogonal FSs.

For $k = 2$, it is necessary to have a triplet of two $FS(6;2,2,2)$s and one $FS(6;3,3)$ which are mutually orthogonal. One such arrangement is obtained from $S_9(6;2,2,2)$, $S_{10}(6;2,2,2)$, and $FS_1(6;3,3)$; a second arrangement may be formed from $FS_1(6;2,2,2)$, $FS_2(6;2,2,2)$, and $FS_7(6;3,3)$; and a third arrangement may be constructed from $FS_5(6;2,2,2)$, $FS_7(6;2,2,2)$, and $FS_6(6;3,3)$. No more than these three arrangements were found from the squares in Tables 1 and 3 for two and three symbols.

For the record, $FS_6(6;3,3)$ is orthogonal to $FS_5(6;2,2,2)$ and to $FS_7(6;2,2,2)$, and $FS_7(6;3,3)$ is orthogonal to $FS_1(6;2,2,2)$, to $FS_2(6;2,2,2)$, and to $FS_5(6;2,2,2)$. $FS_1(6;2,2,2)$ to $FS_8(6;2,2,2)$ are orthogonal to $S_9(6;3,3)$ and to $S_{10}(6;3,3)$. Also, $FS_1(6;3,3)$ to $FS_8(6;3,3)$ are orthogonal to $S_9(6;2,2,2)$ and to $S_{10}(6;2,2,2)$. No other squares were found to be orthogonal.

Construction of r-row by c-column Lattice Rectangle Plans

For $v = 36$, one may form a 2-row by 18-column, a 3-row by 12-column, a 4-row by 9-column, or a 6-row by 6-column lattice rectangle plan. The last one has already been discussed. To form a 2-row by 18-column lattice rectangle plan, one needs two squares with two symbols and two squares with three symbols which form a mutually orthogonal set. Three such arrangements are possible from the above squares; these are:

<u>Plan 1</u>: $S_9(6;3,3)$ used to find the 18 treatments in each row and $FS_1(6;2,2,2)$, $FS_2(6;2,2,2)$, and $FS_7(6;3,3)$ used to form the 18 columns.

<u>Plan 2</u>: $S_{10}(6;3,3)$ used to find the 18 treatments for each row; $FS_5(6;2,2,2)$, $FS_7(6;2,2,2)$, and $FS_6(6;3,3)$ used to form the 18 columns.

<u>Plan 3</u>: $FS_1(6;3,3)$ used to find the 18 treatments for each row; $S_9(6;2,2,2)$, $S_{10}(6;2,2,2)$, and $FS_2(6;3,3)$ used to form the 18 columns.

No more arrangements were found.

For a 3-row by 12-column lattice rectangle plan, two arrangements were found as follows:

<u>Plan 1</u>: $S_9(6;2,2,2)$ is used to find the 12 treatments in each of the three rows; $FS_6(6;3,3)$, $FS_7(6;3,3)$, and $FS_5(6;2,2,2)$ were used to form the 12 columns.

<u>Plan 2</u>: $FS_1(6;2,2,2)$ is used to find the 12 treatments in each of the three rows; $FS_2(6;2,2,2)$, $S_9(6;3,3)$, and $S_{10}(6;3,3)$ are used to form the 12 columns.

No additional plans were found.

For a 4-row by 9-column lattice rectangle, two plans may be constructed as follows:

<u>Plan 1</u>: $S_9(6;3,3)$ and $S_{10}(6;3,3)$ are used to find the nine treatments in each of the four rows; $FS_1(6;2,2,2)$ and $FS_2(6;2,2,2)$ are used to form the nine columns.

<u>Plan 2</u>: $FS_1(6;3,3)$ and $FS_2(6;3,3)$ are used to find the nine treatments which appear in each row; $S_9(6;2,2,2)$ and $S_{10}(6;2,2,2)$ are used to form the nine columns.

Statistical Analysis

Standard incomplete block response models may be used directly for the incomplete block designs of the second section. Likewise, intra-block and inter-block analyses may be carried out in the same manner. For the r-row by c-column lattice rectangles, again one may use standard lattice square and lattice rectangle response models and analyses. Inter-row and inter-column information may be recovered in the same manner as it is for lattice squares.

Discussion

The procedures discussed herein can be used to obtain incomplete block and lattice rectangle designs for other values of v. The F-square and F-rectangle theory discussed in Federer *et al.* (1984), Hedayat and Federer (1984), Mandeli and Federer (1984), and Schwager *et al.* (1984) would be useful in this context. An investigation of F-squares of order six to find the largest possible number of sets of triplets and quartets of F-squares with two and three symbols, is needed to obtain additional plans for various values of k.

Literature Cited

Federer, W. T. (1950). The general theory of prime-power lattice designs. V. The analysis for a 6 x 6 incomplete lattice square illustrated with an example. *Biometrics* 6, 34-58.

Federer, W. T. (1955). *Experimental Design — Theory and Application*. Macmillan, New York. (Republished by the Oxford IB & H Publishing Co., New Delhi, 1967, 1974.)

Federer, W. T. (1975). On the construction of F-squares and single degree-of-freedom contrasts (Preliminary). Paper No. BU-566-M in the Biometrics Unit Mimeo Series, Cornell University, Ithaca, New York.

Federer, W. T., Hedayat, A., and Mandeli, J. P. (1984). Pairwise orthogonal F-rectangle designs. *J. Statistical Planning and Inference* 10,

Finney, D. J. (1982). Some enumerations for the 6 x 6 Latin squares. *Utilitas Mathematica* 21 A, 137-153.

Hedayat, A. and Federerer, W. T. (1984). Orthogonal (v/2) x 2v F-rectangles for all even v. Bulletin, Calcutta Statistical Association (to appear).

Kempthorne, O. (1952). The Design and Analysis of Experiments. Wiley, New York.

Kempthorne, O. and Federer, W. T. (1948). The general theory of prime-power lattice designs. II. Designs for p^n varieties in blocks of p^s plots, and in squares. Biometrics 4, 109-121.

Khare, Meena and Federer, W. T. (1981). A simple construction procedure for resolvable incomplete block designs for any number of treatments. Biometrical J. 23(2), 121-132.

Kirton, H. C. (1984). Mutually orthogonal partitions of the 6 x 6 Latin squares. Unpublished paper, Statistics, N.S.W. Dept. of Agric., Sydney, N.S.W., Australia, 2006.

Mandeli, J. P. and Federer, W. T. (1984). On the construction of mutually orthogonal F-hyperrectangles. Utilitas Mathematica 25, 315-324.

Patterson, H. D. and Williams, E. R. (1976). A new class of resolvable incomplete block designs. Biometrika 63, 83-92.

Schwager, S. J., Federer, W. T., and Mandeli, J. P. (1984). Embedding cyclic Latin squares of order 2^n in a complete set of orthogonal F-squares. J. Statistical Planning and Inference 10, 207-218.

Yates, F. (1937). The design and analysis of factorial experiments. Imperial Bureau of Soil Science, Tech. Comm. 35.